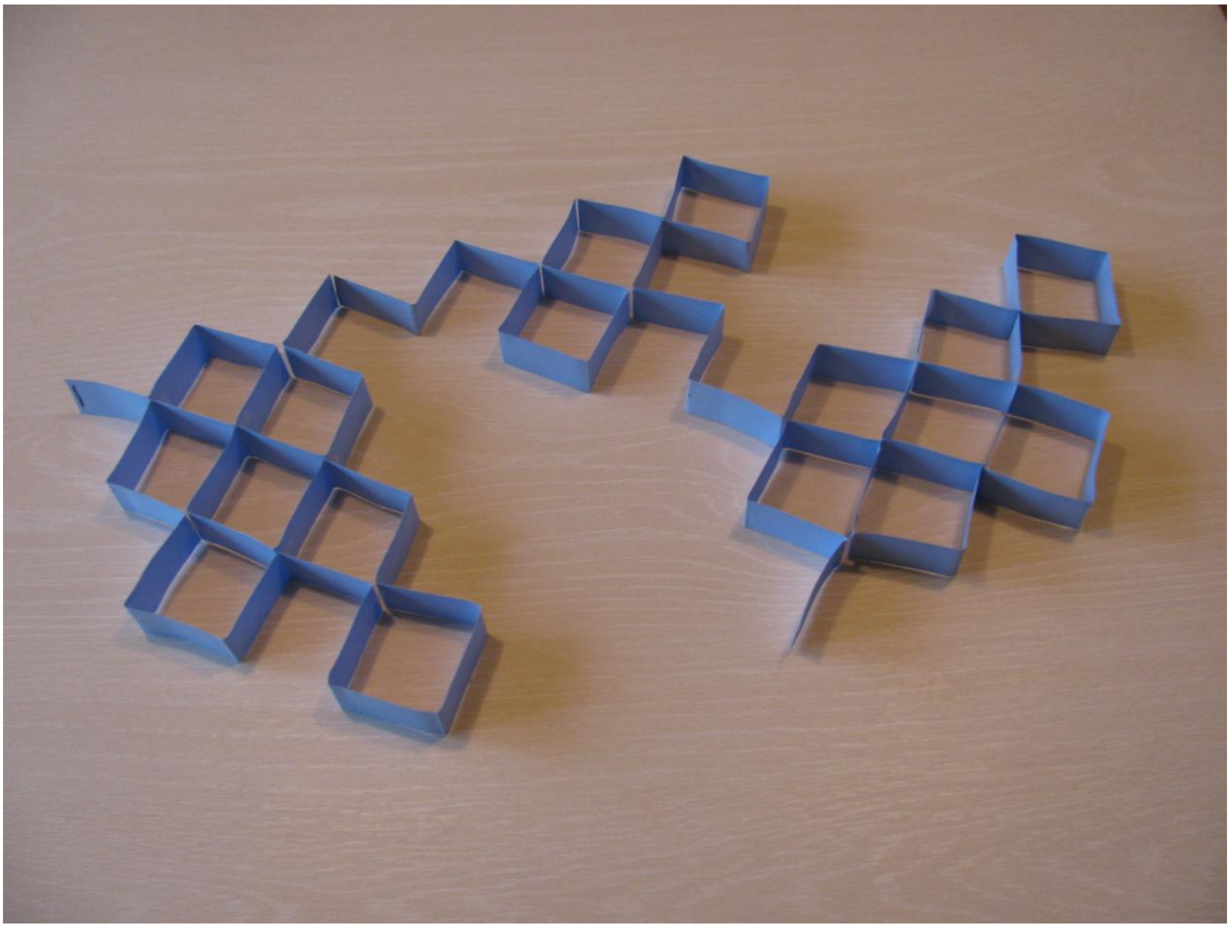
MATHEMATICS B-DAY 2012

Friday November 16, 9:00-16:00 hr



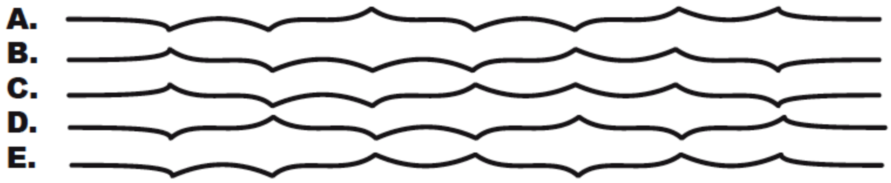
(Cr)easy!

Mathematics B-day is sponsored by

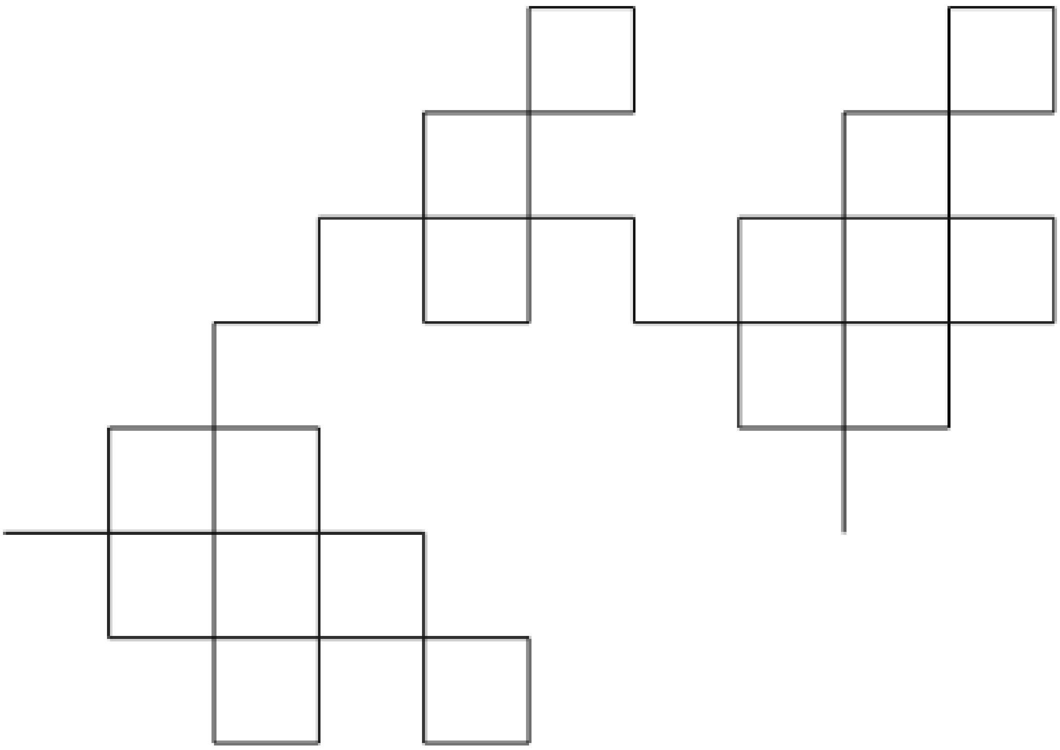
Two appetizers that show the idea of the Math B-day of this year.

Problem 6 of the Kangaroo competition wizPROF 2010:

6. A paper strip is folded in two three times. Then it is unfolded again. When holding the strip upright, you can see the folds from above. Which of the following strips cannot be seen?



In the figure below you can see a labyrinth-like figure that arises from folding a strip of paper in two six times, according to a prescribed folding recipe. It represents in a orderly manner the photo on the front page.
After, the folded strip is unfolded again so that there are 90 degree bends on the folding lines:



The topic of this year's Mathematics B-day will be this kind of labyrinthine patterns that result from repeatedly folding strips of paper and unfolding them to 90 degree bends.

Introduction to the assignment

The structure of this year's Mathematics B-day is very straightforward: the topic is repeatedly folding in two strips of paper in a prescribed manner. Sharp folds will result in an interesting geometrical figure, that consists of straight parts that bend at a right angle on every folding line.
The folding recipe is simple, but it turns out that it gives rise to patterns that quickly end up looking rather complex. You are going to study these patterns this day.

The goal of this assignment is that you can explain the patterns and even (without folding them for real) predict them. Not just for the patterns that you yourself constructed, but also for the patterns that you might get after folding for instance 10 times. And you really can't do those with a strip of paper!

Structure of the assignment

In **Part A** (Exploration) targeted questions will help you on your way with folding and observing the properties of the patterns that emerge. Perform the folds and compare a series of folds with the resulting geometrical figure. This will be of use to support you with your own research in part B.

After this exploration you will be able to go your own way in **Part B** (Research). You can make use of the research suggestions that are given, but don't restrict yourself to these suggestions. There really is a lot more to discover!
What is important is that in your report of the day's activities you explain to an outsider-with-feeling-for-mathematics what you discovered.
You can use the representations that were given in part A, but your own ways of communicating what you discovered are certainly welcome as well.

You will also have available an applet that can be used to make patterns. Don't use the applet too soon! Use the exploration phase in part A to get a good understanding of the folding process and its results by folding strips of paper your self. So don't use the applet until part B.
The applet may come in handy to support your ideas in the elaboration of your project.

Planning your day

- Take at least two hours to steadily explore folding and the resulting patterns. The amount of time you invest on part A does play a significant role in how in-depth your research in part B can be.
- Reserve enough time for your report. It has to be handed in by 16:00 hr. Do not leave starting it until the last!

The final product

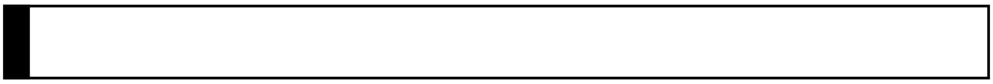
This Mathematics B-day assignment lends itself very well for a report that doesn't slavishly follow questions and suggestions. Make sure that the report can be read independently by someone who doesn't have the assignment at hand.
If you include handwritten items, take care to use a **black pen** to write (and draw) them. The material must be easy to copy legibly.

Enjoy, and good luck!

PART A: EXPLORATION

Introduction

For this Mathematics B-day assignment you need strips of paper: lots of them! The exact measurements are not important. You could for instance length-wise cut an A4 sheet of paper into eight strips. At the end of a strip you put a small black mark on one side. That is the starting point of the strip.



Place the strip on its side on the table in front of you *with the black mark facing you* as drawn above.
Now you will fold the strip a few times, without unfolding it in-between. You always fold the right-hand side of the (folded) strip towards the left-hand side.
After folding like this a few times it looks complicated, but you can always find the starting point again because of the black mark.

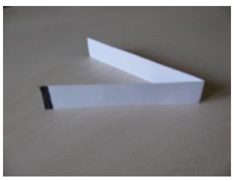
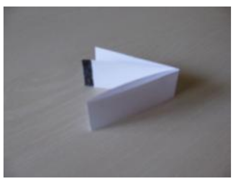
There are always two ways to fold the strip:

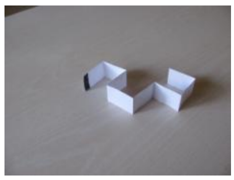
- Folding away from you, so to the left. The right-hand side will end up *behind* the left-hand side. We indicate this with the letter l.

- Folding towards you, so to the right. The right-hand side will end up *in front* of the left-hand side. We indicate this with the letter r.

The folded strip of paper is folded again, and this is repeated a number of times. Each time it can be l or r.

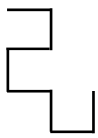
A **folding recipe** is a series of letters l and r. These are used to prescribe the folding actions. You follow the recipe from left to right. An example will clarify this.
There are three folding steps in the folding recipe l r r:

Fold 1: l, so to the left	Fold 2: r, so to the right	Fold 3: r, so to the right
		



After performing the folding recipe you do the following:
- Unfold the strip. Take care to have 90 degree angles on all folding lines and to keep the connecting pieces straight.
- Put the unfolded strip on its side in front of you on the table, with the black mark facing you on the left.

You can see the result on the photo on the right.

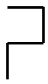
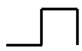
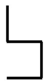


From the top you can see the shape that is drawn on the right. We call this type of figure a **walking pattern**. Mainly we draw the walking pattern so that the start is **from the black mark facing horizontally to the right**.

The walking pattern can also be described as a route from the starting point to the end of the strip past all bends.
Imagine that you are following the walking pattern from start to finish. Each time you walk along a stretch of straight road, followed by a 90 degree bend either to the left or to the right. After the next straight stretch of road, there will be another bend to the left (*L*) or to the right (*R*). Only the bends matter; the stretches in between are all the same length.
The route for the walking pattern in the example is *RRLLRLL.*
For simplicity's sake, we will also call the series of letters *L* and *R* the **walking pattern**, just like the figure itself.

So there are two important points in folding and unfolding the strips:
- the **folding recipe**. This indicates how to fold, using a series of letters *l* and *r*.
- the **walking pattern**. This is the result after following a folding recipe, described in a figure or a series of letters *L* and *R*.

<u>Introductory research</u>
Below are the four folding recipes that fold twice. The illustrations all start horizontally to the right, but that is just to show where and how to start.

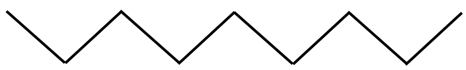
folding recipe	walking pattern	
$r\ r$		RRL
$r\ l$		LRR
$l\ r$		RLL
$l\ l$		LLR

Of course you check the four possible patterns by performing the folding recipes and looking carefully at the results and how they arose.

To further familiarise yourself with folding recipes and their walking patterns, you will first perform all folding recipes for three folds.

> It is useful for reasoning about folding recipes and their walking patterns to first perform a number of concrete examples yourself with strips.
> *So do not skip these hands-on activities!*

♦ **Exploratory question 1.**
a. There are eight possibilities for a three-fold folding recipe. What are they?
b. Make an overview for the three-fold recipes, like the one above for two-fold recipes: both the folding recipes and their walking patterns.
c. You won't find the zigzag pattern below as a possible three-fold pattern. You could have known in advance. How?



Folding twice always results in the same number of straight pieces (4) and the same number of bends (3).

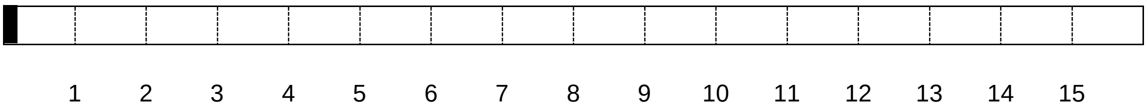
♦ **Exploratory question 2.**

a. How many straight pieces and how many bends does a walking pattern have after folding three times?

b. How many straight pieces and how many bends does a walking pattern have after folding *n* times (*n* = 1, 2, 3, 4, 5, ...)?

The order of the bends

Four folds will give you 16 straight pieces and 15 bends after unfolding.
Below, the dotted folding lines have been numbered left to right in the completely unfolded strip. These are the positions on the strip where bends will appear in the walking pattern.



♦ **Exploratory question 3.**

a. In which position (which number in the drawing above) will you find the folding line for the initial fold?
And which positions (numbers) will indicate the second, third and fourth fold?

There is an *L* or an *R* in each position, depending on the folding recipe. Someone has made a walking pattern with an *R* in the positions 2, 7, 8 and 12. That is enough information to retrieve the whole walking pattern!

b. Determine the whole walking pattern (give the series of *R*s and *L*s).

c. What was the folding recipe?

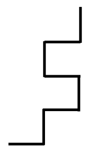
♦ **Exploratory question 4.**

We take a six-fold folding recipe.
This gives you a walking pattern with 64 straight pieces and 63 bends

a. Extra information: the first bend from the left is an *R*.
For which of the remaining 62 bends can you now be sure that it is an *R* or an *L*?

b. Some more information: the walking pattern starts with *RLLR.*
How many bends are left for which you cannot decide yet whether they are *R* or *L*?

♦ **Exploratory question 5.**



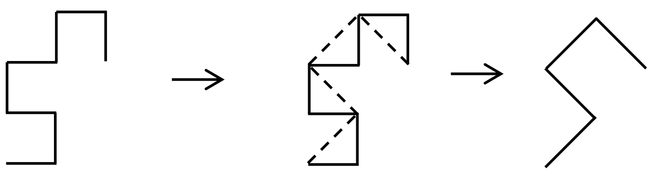
a. Why can the figure on the right not be the result of a folding recipe?

b. Why can the series *RLLRRRLLRLLRRRL* not occur as a walking pattern?

The connection between two walking patterns

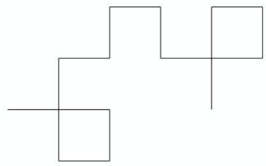
In a given walking pattern, bends are connected by skipping one bend each time. We start from the left (start).

Look at the dotted line in the middle picture. It goes from the start to the end of the walking pattern and skips one bend each time. This gives us the result on the right. Four bends in the walking pattern have been straightened as it were.



- **Exploratory question 6.**
  - **a.** The walking pattern in the picture on the left goes with the folding recipe $r\ l\ l$.
    Explain why the picture on the right is a walking pattern as well. For which folding recipe?
  - **b.** Is there another left-hand walking pattern that results in the same walking pattern on the right?

- **Exploratory question 7.**
  The walking pattern drawn here goes with the folding recipe $r\ l\ l\ r$.



  - **a.** What will change in the walking pattern if you replace the first r in the folding recipe with an l?
    What changes do you see in the letter code for the walking pattern?
  - **b.** What will change in the original walking pattern if you replace the final r in the folding recipe by an l?
    What changes do you see now in the letter code of the walking pattern?

- **Exploratory question 8.**
  The walking pattern *LRRLLLRLLRRRLLR* is the result of a four-fold folding recipe.
  - **a.** There is a folding recipe where the *R* in the second position is an *L* and otherwise as many letters as possible remain the same as in the given walking pattern.
    Which letters in the given walking pattern have to change and what change in the folding recipe will then have to take place?
  - **b.** There is a folding recipe as well where the *L* in position 4 of the given walking pattern is an *R* and otherwise as many letters as possible remain the same.
    What can you say now about the letter code and the folding recipe?

<u>Schematic overview about how a walking pattern emerges from a folding recipe</u>

Below is a schematic representation of how the final walking pattern emerges step by step while performing the folding recipe $r\ l\ r\ l$
From bottom to top you see the walking pattern (with *L* and *R*) after one, two, three and four folds. Every time the position where the bends end up in the finished walking pattern (positions 1 to 15) are given.

after fold 4	*L*	*R*	*R*	*L*	*L*	*L*	*R*	*R*	*L*	*R*	*R*	*R*	*L*	*L*	*R*
after fold 3		*R*		*L*		*L*		*R*		*R*		*R*		*L*	
after fold 2				*L*				*R*				*R*			
after fold 1								*R*							
position	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

- **Exploratory question 9.**
  Three bends are given for a specific four fold folding recipe.

after fold 4			*L*	*R*		*R*				**?**		**?**	**?**		
after fold 3															
after fold 2															
after fold 1															
position	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Which letters (*L* or *R*) are in the three positions with the question marks?

PART B: YOUR OWN RESEARCH

You have already done a lot of experimental research in the exploratory questions in part A by folding strips yourself. Now use these experiences to dive deeper into the secrets of the walking patterns that emerge from various folding recipes.

Deal
Use the applet to verify any conjectures you have. Only mentioning a number of examples is not enough proof that your conjecture is always true. Therefore always try to find sound arguments to prove your conjecture.

Three statements

Below are three true statements, along with a question. Answering the questions in a convincing way may help you with your research. Of course, statements about your own discoveries are also welcome.

- *Statement 1*:
  For every walking pattern, the first and the last straight piece are perpendicular to each other.
  Question:
  Try to find a solid reasoning that shows this is always true.

- *Statement 2*:
  For a given folding recipe you can predict for every bend of the accompanying walking pattern whether it is *L*(eft) or *R*(ight).
  Question:
  How can you predict for the folding recipe *r r r l r r r l r* what kind of bend (*R* or *L*) there will be in position 16?

- *Statement 3*:
  For any given walking pattern you can always find out the accompanying folding recipe.
  Question:
  Can you give a general method that allows you to find the accompanying folding recipe from any given walking pattern?

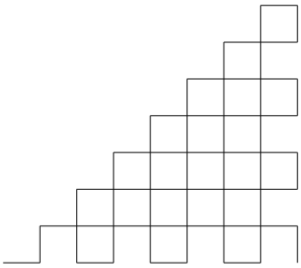
Below, some research suggestions are given. Select among the suggestions or explore your own research questions.
Describe clearly what you investigated and what your findings were.

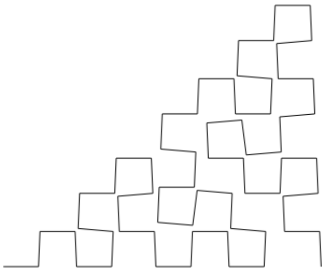
- Select a fixed number n (= 1, 2, 3, 4, 5, ...).
  For all walking patterns that you can make using the chosen n folds, the distance from start to finish is the same.
  Investigate why this is so.
  Note: that distance of course depends on the length of the strip you start with. It makes sense to start with a strip of length 2^n, so that all straight pieces will have length 1.

- Select a fixed number n (= 1, 2, 3, 4, 5, ...).
  Mark the starting point of a walking pattern in a fixed place on a sheet of grid paper.
  - Where are the possible end points of a walking pattern after n folds now?
  - On the way from the starting point to the end point there may be points that are further away from the starting point than the end point is. Investigate what distances are possible. Can you express the maximum distance from the starting point to any point in the walking pattern in n?

- There are many walking patterns where the folding points in the pattern touch. This means that you revisit an earlier point. This occurs when RRR or LLL appears in the walking pattern: after three bends *R* or *L* you are back at the starting point. If n = 5, each walking pattern will have at least one sub-series *LLL* (you can verify this)
  - Show that for every $n > 5$ each walking pattern contains at least one sequence *LLL*.
  - Argue that the number of sequences *LLL* increases (or at least does not decrease) for increasing n in each possible walking pattern.

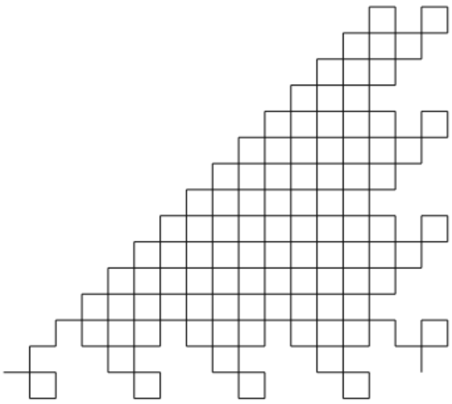
- Study the figures below: on the left a walking pattern with 90 degree bends, on the right the same pattern with bends that are slightly less than 90 degrees. The figure on the right clearly shows that there is no overlap in the walking pattern: a straight piece is never traversed twice. So *RRRR* or *LLLL* cannot occur in a walking pattern.





- - Can you prove that *RRRR* (or *LLLL*) is not possible in a walking pattern?
  - When do you run into points of contact that only occur later in the pattern, like in the walking pattern given here?

- In a walking pattern you can find symmetries, as can be seen in the next picture. That is true for the whole picture, but also for smaller sections within it. Can you reason out where these different kinds of symmetry occur?



- Take a connected block of 2^k letters *L* and *R* from a given walking pattern. The block of letters will have a length of 4, 8, 16, 32, ... ($k \geq 2$).
  Now, in that connected block count the number of times that *L* occurs and the number of times that *R* occurs. The difference between the two amounts turns out to be 0 or 2.
  - Can you explain that?

  We call the difference between the number of *R* and the number of *L* in a block the *discrepancy* of that block. The previous statement says that at least these special blocks of length 2^k will have a discrepancy 0 or 2.
  - Investigate what the maximum discrepancy is in a walking pattern for n folds and formulate a conjecture.

The end